On the Eisenstein classes of Hilbert-Blumenthal modular varieties

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The sheaf theory of polylogarithms, first developed for the multiplicative group, provides an interpretation of special values of the zeta function in terms of Hodge theory (Part 1). Such a theory of polylogarithms exists also for complex abelian schemes (Part 2). For elliptic curves the objects of this theory have been intensely studied (see e.g. Results R1, R2 and R3 below). In the higher dimensional case, some results have been proven (see e.g. Results R1', R2'), but no link between this theory and some special values of L-functions was known. Specializing the geometric context to Hilbert-Blumenthal modular families of abelian varieties, we establish such a link (see Result R3') and obtain a geometric proof of the Klingen-Siegel Theorem (Part 3).

1. Review of the classical case

Beilinson's conjectures hold for Spec($\mathbb{Q}$) (Borel, Rapoport, ...) and one may interpret this result as follows. The subspace $\zeta(3)\mathbb{Q}$ of $\mathbb{R} = \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(3))$, where $\mathrm{MHS}_{\mathbb{R}}$ denotes the category of polarizable real mixed Hodge structures, compares extensions of motives and extensions of real mixed Hodge structures. Thus

$$(*) \qquad \zeta(3)\mathbb{Q} \text{ is a canonical subspace of } \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(3)).$$

The sheaf theory of polylogarithms for $\mathbb{G}_m$ (Beilinson, Deligne, Ramakrishnan) provides an explanation of the assertion $(*)$ using only Hodge theory. Let VMHS(X) be the category of admissible polarizable variations of rational mixed Hodge structures over X, for X a smooth complex algebraic variety. The objects of this theory are

- the logarithm (a pro-object of $\mathrm{VMHS}(\mathbb{G}_{m,\mathbb{C}})$), denoted by $\mathcal{L}og$,
- the polylogarithm (an element of $\mathrm{Ext}^1_{\mathrm{VMHS}(\mathbb{G}_{m,\mathbb{C}}\setminus\{1\})}(\mathbb{Q}(0), \mathcal{L}og_{|\mathbb{G}_{m,\mathbb{C}}\setminus\{1\}})$,
- the Eisenstein classes (elements of $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathbb{Q}(k))$ for some $k \geq 0$, where MHS denotes the category of polarizable rational mixed Hodge structures).

All of these objects can be described explicitly, e.g. the polylogarithm corresponds to a pro-matrix in which appear all the multivalued functions Li_k, $k \geq 1$. It turns out that the Eisenstein classes are related to some special values of the zeta function and this provides the desired explanation of the assertion $(*)$ using only Hodge theory.

2. The abelian case

This sheaf theory of polylogarithms is defined in a more general geometric setting (cf. [13]) and gives some special elements (the Eisenstein classes) which should have remarkable properties. For instance, such a theory exists for complex abelian schemes.

Fix a smooth complex algebraic variety S and a complex abelian scheme $\pi\colon A \to S$ of pure relative dimension g. Let U be the complement of the zero section and let $\mathcal{H} := (R^1\pi_*\mathbb{Q})^\vee$ (polarizable variation of rational pure Hodge structures of weight -1 over S). For X a smooth complex algebraic variety, we denote by $\mathrm{MHM}(X)$ the category of algebraic mixed Hodge modules over X and we recall that one can see $\mathrm{VMHS}(X)$ as a full subcategory of $\mathrm{MHM}(X)$ in a canonical way. As in the case of the mupltiplicative group, one can define (see e.g. Sections 3–5 of [4])

- the logarithm (a pro-object of $\mathrm{VMHS}(A)$), denoted by $\mathcal{L}og$,
- the polylogarithm (an element of $\mathrm{Ext}^{2g-1}_{\mathrm{MHM}(U)}((\pi^*\mathcal{H})_{|U}, \mathcal{L}og_{|U}(g))$,
- the Eisenstein classes (elements of $\mathrm{Ext}^{2g-1}_{\mathrm{MHM}(S)}(\mathbb{Q}(0), (\mathrm{Sym}^k\mathcal{H})(g))$ for some $k \geq 0$).

For the elliptic case ($g = 1$), the definition and the study of these objects are due to Beilinson and Levin. For the universal elliptic curve over the modular curve, we have the following properties. We refer the reader to [1] for precise formulations and proofs (see also [11] for R3).

- R1 The Eisenstein classes have a motivic origin.
- R2 The polylogarithm is a 1-extension of admissible polarizable *variations* of rational mixed Hodge structures which can be explicitly described by a pro-matrix in which appear the Debye polylogarithms.
- R3 The residues of the Eisenstein classes at the ∞ cusp of the modular curve are related to some values of Bernoulli polynomials.

Later the definitions have been extended to any complex abelian schemes (this follows from the content of [13]) and the following results have been proven.

- R1' The Eisenstein classes have a motivic origin (see [6]).
- R2' The currents constructed by Levin in [9] provide an explicit description of the polylogarithm at the topological level. This result had been conjectured by Levin and is announced in the Note [2]. We refer to [4] for a proof (see the proof of Théorème 4.5 and Corollaire 4.7 in loc. cit.). We note that if the relative dimension of the abelian scheme is greater than 2, the polylogarithm is *not* an extension of admissible polarizable variations of rational mixed Hodge structures (cf. Theorem III-2.3 b) of [13]).

3. The Hilbert-Blumenthal case

If one specializes the geometric setting to Hilbert-Blumenthal modular families of abelian varieties, we show, using the result R2', the following generalization of the result R3.

R3' The Eisenstein classes degenerate at the ∞ cusp of the Baily-Borel compactification of the base in special values of an L-function associated to the underlying totally real number field. This result is stated in the Note [3] and the reader may consult [5] for a proof (see the proof of Théorème 5.2 in loc. cit.).

We mention that there exists a different proof of the result R3' (see [7]). Since the residues at the ∞ cusp are rational numbers, we can deduce from the result R3' the Klingen-Siegel Theorem. We note that our proof presents some analogy with the original one (cf. [8]). We also point out that there exist two other proofs due to Sczech [12] and Nori [10] which use rational cohomology classes to deduce the Theorem. Thus our proof has also some analogy with both of them.

References

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