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Differentiable stacks and Lie groupoids

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0 Introduction

In january 2007, I gave a talk about the link between the differentiable stacks and the Lie groupoids during the workshop on orbifolds which held at the University of Pardeborn. I explained how one can associate to a Lie groupoid G a differentiable stack BG consisting of the principal G-bundles and to a generalized morphism of Lie groupoids $f\colon G \to G'$ a morphism of stacks $B(f)\colon BG \to BG'$. At the end, I stated two questions:

Q1. Is B a morphism of (2-)categories ?

Q2. If it is, does B induce an equivalence of (2-)categories ?

In the article [LTX, Rem 2.6], positive answers are given and some elements of proof can be found in [BX] (cf. [BX, Dictionnary Lemmas, Part 2.6]). This document is devoted to the study of these two questions. Our work is based on the articles of Behrend, Xu [BX], Heinloth [H], Laurent-Gengoux, Tu and Xu [LTX], Naumann [N], Pronk [P].

We now give some details on the plan and on the results.

1 We explain the notion of principal G-bundles over a manifold for a Lie groupoid G. This is a generalization of the corresponding notion when G is a Lie group. We follow very closely the exposition of Moerdijk and Mrčun [MM, Chap 5].

2-3 We recall some basic properties of stacks and explain how one can descend a morphism between two atlases to a morphism between the corresponding differentiable stacks. The results of these two parts are well known results for algebraic stacks and we refer the reader to the book of Laumon and Moret-Bailly [LM-B] for the proofs.

4-8 We define the 2-categories of Lie groupoids $\mathcal{LG}$ and rigidified differentiable stacks $\mathcal{RDS}$ (an object of $\mathcal{RDS}$ is a differentiable stack with a fixed atlas) and we construct two 2-functors $A\colon \mathcal{RDS} \to \mathcal{LG}$ and $B'\colon \mathcal{LG} \to \mathcal{RDS}$. We note that the definition of B' on the level of 1-morphisms differs from the one given during the talk. The comparison of the two constructions is left to further investigations. We prove the following

Theorem (8.1) – *The 2-functors A and B' are inverse equivalences of 2-categories.*

These five parts are translations from the algebraic stacks to the differentiable stacks of the results of Naumann [N, 3.1,3.2,3.3].

9 On the one hand many properties of Lie groupoids are invariant under Morita equivalences and on the other hand one may be interested in the 2-category of differentiable stacks $\mathcal{DS}$ instead of $\mathcal{RDS}$. Note that we have an obvious forgetful 2-functor $For\colon \mathcal{RDS} \to \mathcal{DS}$ and that every Lie groupoid is Morita equivalent to an étale Lie groupoid. The main result of this part is the

Theorem (9.2) – *The 2-functor $For \circ B'$ induces a morphism of bicategories $B\colon \mathcal{ELG}[W^{-1}] \to \mathcal{DS}$ which defines an equivalence of bicategories.*

Here $\mathcal{ELG}[W^{-1}]$ is the localisation of the 2-category of étale Lie groupoids $\mathcal{ELG}$ w.r.t. the class of weak equivalences W in the sense of Pronk [P, Part 2]. We point out that Pronk defines also an equivalence of bicategories between $\mathcal{ELG}[W^{-1}]$ and $\mathcal{DS}$ via the 2-category of differentiable étendues [P, Cor 42].

10-11 The bicategory $\mathcal{ELG}[W^{-1}]$ can be described using the notion of generalized morphisms. In this part we prove this fact, i.e. the

Theorem (11.2) – *We have a canonical morphism of bicategories $C' \colon \mathcal{ELG} \to \mathcal{ELG}^+$ which induces an equivalence of bicategories $C \colon \mathcal{ELG}[W^{-1}] \to \mathcal{ELG}^+$.*

12 In this last part we use the previous results to deduce an equivalence of categories (see Thm 12.1). In fact we divide both $\mathcal{ELG}[W^{-1}]$ and $\mathcal{DS}$ by their 2-cells. As a consequence we can not consider anymore 2-morphisms in $\mathcal{DS}$. This leads to a notion which is not so easy to handle.

Acknowledgments: I am very glad to thank Stefan Wolf for interesting discussions on differentiable stacks and Patrick Schützdeller for indicating me the article of Pronk [P] and explaining me some points on Lie groupoids. I thank also Torsten Wedhorn for his help during the preparation of the talk and the redaction of this document. In particular he suggested me that the article of Naumann [N] could be helpful to answer the questions Q1 and Q2.

Notations:

$\mathcal{M}$	the category of manifolds,
$\mathcal{G}$	the 2-category of groupoids,
YLS	Yoneda Lemma for stacks (cf. [H, Lem 1.3]),
$\star$	horizontal composition of 2-morphisms in a 2-category.

1 Principal G-bundles over a manifold for a Lie groupoid G

Let

$G := (G_0, G_1, s, t, e, m, i)$	be a Lie groupoid,
M	be a manifold,
(H, e, m, i)	be a Lie group,
$\mathcal{H} := (\bullet, H, H \to \bullet, H \to \bullet, e, m, i)$	be the Lie groupoid associated to H.

We recall the following definitions.

Definition 1.1 *[MM, p. 125] – A right action of G on M along a smooth map $\varepsilon \colon M \to G_0$ is given by a smooth map*

$$\mu \colon M \underset{\varepsilon, G_0, t}{\times} G_1 \to M, \quad (m, g) \mapsto mg$$

such that

a) $\varepsilon(mg) = s(g)$ *for all* $m \in M$, $g \in G_1$, *such that* $\varepsilon(m) = t(g)$,

b) $m\, e(\varepsilon(m)) = m$ *for all* $m \in M$,

c) $(mg)g' = m(gg')$ *for all* $m \in M$, $g, g' \in G_1$, *such that* $s(g) = t(g')$, $\varepsilon(m) = t(g)$.

A left action of G on M along a smooth map $\varepsilon\colon M \to G_0$ is defined analogously.

Remark 1.2 – *A right action of $\mathcal{H}$ on M along $M \to \bullet$ is nothing but that a right action of H on M.*

We now define the notions of G-bundles over M and morphisms between them.

Definition 1.3 *[MM, p. 144-145] –*

i) A G-bundle over M is a manifold P equipped with two smooth maps

$$\begin{array}{ccc} P & \xrightarrow{\varepsilon} & G_0 \\ {\scriptstyle\pi}\downarrow & & \\ M & & \end{array}$$

and a right action of G along ε,

$$\mu\colon P \underset{\varepsilon, G_0, t}{\times} G_1 \to P, \quad (p,g) \mapsto pg$$

such that $\pi(pg) = \pi(p)$ for all $p \in P$, $g \in G_1$ such that $\varepsilon(p) = t(g)$.

ii) Let (P,π,ε,μ) and $(P',\pi',\varepsilon',\mu')$ be two G-bundles over M. A morphism of G-bundles over M from (P,π,ε,μ) to $(P',\pi',\varepsilon',\mu')$ is a smooth map $f\colon P \to P'$ which commutes with all the structure maps, i.e.

a) $\pi(p) = \pi'(f(p))$ for all $p \in P$,

b) $\varepsilon(p) = \varepsilon'(f(p))$ for all $p \in P$,

c) $f(pg) = f(p)g$ for all $p \in P$, $g \in G_1$ such that $\varepsilon(p) = t(g)$ ($= \varepsilon'(f(p))$ according to b).

Remark 1.4 *[BX, Rem p. 11] – Behrend and Xu give the following interpretation of a G-bundle over a manifold M. An element p of P can be viewed as an arrow from $\pi(p)$ to $\varepsilon(p)$. The action of G on M along ε corresponds then to the composition of such arrows.*

Remark 1.5 – *The morphisms of G-bundles over M can be composed in an obvious way. So we have defined a category which is called the category of G-bundles over M.*

Remark 1.6 – *The functor*

$$\begin{array}{ccc} \mathcal{H}\text{-bundles over } M & \to & H\text{-bundles over } M, \\ (P,\pi,\varepsilon,\mu) & \mapsto & (P,\pi,\mu) \end{array}$$

where μ in the right hand side is viewed as a H-action on P (cf. Remark 1.2), and the functor

$$\begin{array}{ccc} H\text{-bundles over } M & \to & \mathcal{H}\text{-bundles over } M, \\ (P,\pi,\mu) & \mapsto & (P,\pi,P \to \bullet,\mu) \end{array}$$

where μ in the right hand side is viewed as a right action of $\mathcal{H}$ on P along $P \to \bullet$ (cf. Remark 1.2), are mutually inverse to each other. So the category of $\mathcal{H}$-bundles over M and the category of H-bundles over M are isomorphic.

In the sequel we will focus on a full subcategory of the category of G-bundles over M. In order to define its objects we introduce the following definition.

Definition 1.7 *[MM, p. 145] – A G-bundle $(P, \pi, \varepsilon, \mu)$ over M is said to be principal if*

a) *π is a surjective submersion,*

b) *the map $(pr_1, \mu)\colon P \underset{\varepsilon,G_0,t}{\times} G_1 \to P \underset{\pi,M,\pi}{\times} P, \quad (p, g) \mapsto (p, pg)$ is a diffeomorphism.*

Example 1.8 *– The manifold G_1 equipped with the two maps*

$$\begin{array}{ccc} G_1 & \xrightarrow{s} & G_0 \\ {\scriptstyle t}\downarrow & & \\ G_0 & & \end{array}$$

and the right action of G along s

$$m\colon G_1 \underset{s,G_0,t}{\times} G_1 \to G_1$$

given by the multiplication of G is a principal bundle over G_0 which is called the unit bundle of G and denoted by $U(G)$.

Example 1.9 *– Let $\mathcal{P} := (P, \pi, \varepsilon, \mu)$ be a principal bundle over M and let $f\colon N \to M$ be a smooth map. The manifold $N \underset{f,M,\pi}{\times} P$ equipped with the maps*

$$\begin{array}{ccccc} N \underset{f,M,\pi}{\times} P & \xrightarrow{pr_2} & P & \xrightarrow{\varepsilon} & G_0 \\ {\scriptstyle pr_1}\downarrow & & & & \\ N & & & & \end{array}$$

and the right action of G along $\varepsilon \circ pr_2$ given by

$$Id_N \times \mu\colon N \underset{f,M,\pi\circ pr_1}{\times} (P \underset{\varepsilon,G_0,t}{\times} G_1) = (N \underset{f,M,\pi}{\times} P) \underset{\varepsilon\circ pr_2,G_0,t}{\times} G_1 \to N \underset{f,M,\pi}{\times} P$$

defines a principal G-bundle over N which we denote $f^*\mathcal{P}$. Here all the fiber products are well defined since π is a submersion.

We introduce the category $BG(M)$ defined as the full subcategory of the category of G-bundles over M characterized by

$$Ob(BG(M)) = \text{the class of all principal } G\text{-bundles over } M.$$

This category is a groupoid according to the following lemma.

Lemma 1.10 *[MM, Remarks 5.34 (4) and (5)] – A morphism between two principal G-bundles over M is an isomorphism.*

2 On the 2-category of stacks

We refer to [H, Rem 1.2.4] for the definition of the 2-category of stacks which we denote $\mathcal{S}$.

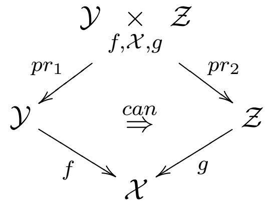
2.1 The universal property of the fiber product in the 2-category of stacks

In this part, we state the universal property of the fiber product in $\mathcal{S}$ as defined in [H, Def 2.1].

Let $f\colon \mathcal{Y} \to \mathcal{X}$ and $g\colon \mathcal{Z} \to \mathcal{X}$ be two 1-morphisms of stacks. Then the stack $\mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z}$ is equipped with two canonical 1-morphims

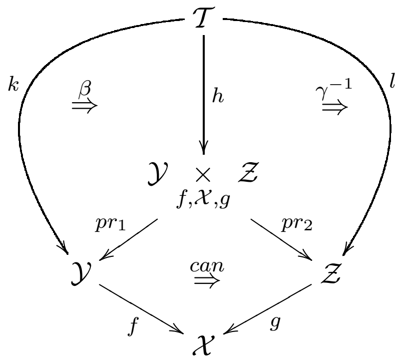
$$pr_1\colon \mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z} \to \mathcal{Y} \quad \text{and} \quad pr_2\colon \mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z} \to \mathcal{Z}$$

and a canonical 2-morphism can



and the following universal property holds.

Proposition 2.1 – *For all triples* $(k\colon \mathcal{T} \to \mathcal{Y},\ l\colon \mathcal{T} \to \mathcal{Z},\ \alpha\colon f \circ k \Rightarrow g \circ l)$ *there exist a unique triple* $(h\colon \mathcal{T} \to \mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z},\ \beta\colon pr_1 \circ h \Rightarrow k,\ \gamma\colon pr_2 \circ h \Rightarrow l)$ *such that the composition of the 2-morphisms in the diagram*



is equal to α.

Proof: Straightforward. □

Remark 2.2 – *It follows from Proposition 2.1 that* $(\mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z}, pr_1, pr_2, can)$ *is a fiber product in the 2-category* $\mathcal{S}$ *and we should note* $\mathcal{Y} \overset{2}{\underset{f,\mathcal{X},g}{\times}} \mathcal{Z}$ *instead of* $\mathcal{Y} \underset{f,\mathcal{X},g}{\times} \mathcal{Z}$.

2.2 Monomorphisms, epimorphisms ans isomorphisms

We define the notion of monomorphism and of epimorphism in the category $\mathcal{S}$. These are translations of the corresponding definitions in the category of algebraic stacks.

Definition 2.3 *[LM-B, p.15] – A monomorphism in* $\mathcal{S}$ *is a monomorphism in the category of 2-functors from* $\mathcal{M}$ *to* $\mathcal{G}$.

Definition 2.4 *[LM-B, Def 3.6] – Let $F\colon \mathcal{X} \to \mathcal{X}'$ be a 1-morphism of stacks. We say that F is an epimorphism in $\mathcal{S}$ if and only if the following condition holds. For all manifolds M', $P' \in Ob(\mathcal{X}'(M'))$, there exists $f\colon M \to M'$ a surjective submersion and $P \in Ob(\mathcal{X}(M))$ such that $F_M(P) \simeq f^*P'$.*

Remark 2.5 *– Let $f\colon X \to Y$ be a smooth map between manifolds. Then the associated morphism in $\mathcal{S}$ between the associated stacks is an epimorphism if and only if f is a surjective submersion.*

Then we state two properties. These are analog of known results in the category of algebraic stacks (see [LM-B, Cor 3.7.1 and Prop 3.8.1]). The proofs work in an analougous way and we leave the details to the reader.

Proposition 2.6 *– A 1-morphism in $\mathcal{S}$ which is a monomorphism and an epimorphism is an isomorphism.*

Proposition 2.7 *– Let us consider the following 2-cartesian diagram of morphisms of stacks.*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\;y\;} & \mathcal{Y}' \\ {\scriptstyle f}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle f'} \\ \mathcal{X} & \xrightarrow[\;x\;]{} & \mathcal{X}' \end{array}$$

i) If f' is an epimorphism, then so is f.

ii) If f and x are epimorphisms, then so is f'.

3 Constructions of morphisms in the 2-category of differentiable stacks

The category of differentiable stacks $\mathcal{DS}$ is the full sub-2-category of $\mathcal{S}$ whose objects are the differentiable stacks. We give two propositions which we will use below two construct 1-morphisms and 2-morphisms between stacks. These are the analogs of the properties for differentiable stacks of properties stated for algebraic stacks in the proof of Proposition 4.18 in [LM-B] (see also [BX, p. 10] and [H, Lem 2.27]).

Consider differentiable stacks $\mathcal{X}$, $\mathcal{X}'$ and let $p\colon X \to \mathcal{X}$ be an atlas of $\mathcal{X}$. From p and a descent datum we can construct a canonical 1-morphism $\mathcal{X} \to \mathcal{X}'$.

Proposition 3.1 *– If $f\colon X \to \mathcal{X}'$ is a morphism of stacks and if α is a 2-morphism in $\mathcal{S}$*

$$\alpha\colon (f \circ pr_1\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}') \Rightarrow (f \circ pr_2\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}')$$

which satisfies the following cocycle condition on $X \underset{p,\mathcal{X},p}{\times} X \underset{p,\mathcal{X},p}{\times} X$

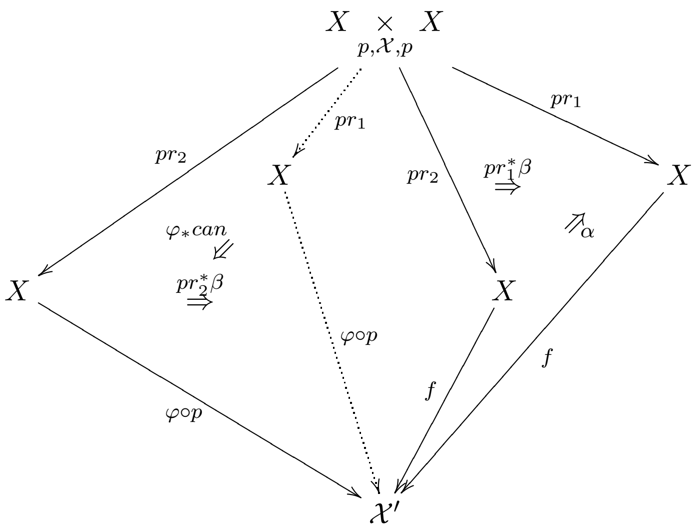
$$pr_{23}^*\alpha \star pr_{12}^*\alpha = pr_{13}^*\alpha \tag{1}$$

then there exist a morphism of stacks $\varphi\colon \mathcal{X} \to \mathcal{X}'$ and a 2-morphism β

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \mathcal{X}' \\ {\scriptstyle p}\searrow & \Uparrow\beta & \nearrow{\scriptstyle \varphi} \\ & \mathcal{X} & \end{array}$$

such that the following diagram 2-commutes

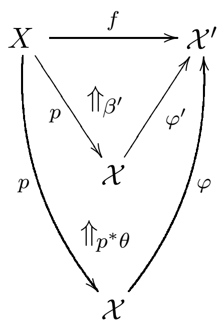
$D(\varphi, \beta)$



i.e. $\alpha \star pr_2^*\beta \star \varphi_* can = pr_1^*\beta$*, where* can *is the structural 2-morphism of the fiber product* $X \underset{p,\mathcal{X},p}{\times} X$*. Moreover, this pair* (φ, β) *is unique up to a unique 2-morphism, i.e. if*

$$(\varphi' \colon \mathcal{X} \to \mathcal{X}', \beta' \colon \varphi' \circ p \Rightarrow f)$$

is another pair such that the diagram $D(\varphi', \beta')$ *commutes then there exists a unique 2-morphism* $\theta \colon \varphi \Rightarrow \varphi'$ *such that the composition of the 2-morphisms in the diagram*



is equal to β*.*

Proof : We refer to [H, p. 10] for the construction of f. This is done using essentially the gluing conditions for stacks (cf. [H, Def 1.1 1,2]). In fact the whole proposition is a consequence of these axioms. □

Proposition 3.2 *– Let* $f, f' \colon X \to \mathcal{X}'$ *be two 1-morphisms of stacks and let*

$$\alpha \colon (f \circ pr_1 \colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}') \Rightarrow (f \circ pr_2 \colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}')$$

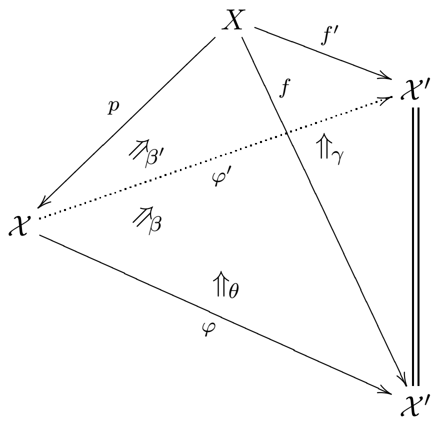
$$\alpha' \colon (f' \circ pr_1 \colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}') \Rightarrow (f' \circ pr_2 \colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}')$$

be two 2-morphisms in $\mathcal{S}$ *which satisfy the cocycle condition (1) on* $X \underset{p,\mathcal{X},p}{\times} X \underset{p,\mathcal{X},p}{\times} X$*. Then we associate to* (f, α) *(resp.* (f', α')*) a morphism of stacks* $\varphi \colon \mathcal{X} \to \mathcal{X}'$ *(resp.* $\varphi' \colon \mathcal{X} \to \mathcal{X}'$*) and a 2-morphism*

$\beta\colon \varphi\circ p \Rightarrow f$ (resp. $\beta'\colon \varphi'\circ p \Rightarrow f'$) using the previous Proposition. Let $\gamma\colon f \Rightarrow f'$ be a 2-morphism in $\mathcal{S}$ such that the diagram

$$\begin{array}{ccc} f\circ pr_1 & \overset{\alpha}{\Longrightarrow} & f\circ pr_2 \\ {\scriptstyle pr_1^*\gamma}\Downarrow & & \Downarrow{\scriptstyle pr_2^*\gamma} \\ f'\circ pr_1 & \underset{\alpha'}{\Longrightarrow} & f'\circ pr_2 \end{array}$$

commutes. Then there exists a unique 2-morphism in $\mathcal{S}$, $\theta\colon \varphi \Rightarrow \varphi'$, such that the diagram



2-commutes.

Proof : This is also a consequence of the gluing conditions for stacks. □

4 The 2-category of rigidified differentiable stacks

The 2-category of rigidified differentiable stacks is defined as follows.

0. The objects are differentiable stacks equipped with an atlas (cf. [H, Def 2.3]), i.e. the pairs $(\mathcal{X}, p\colon X \to \mathcal{X})$ where $\mathcal{X}$ is a differentiable stack and p is an atlas (cf. [H, Def 2.3]) of $\mathcal{X}$.

1. A 1-morphism from $(\mathcal{X}, p\colon X \to \mathcal{X})$ to $(\mathcal{X}', p'\colon X' \to \mathcal{X}')$ is a pair of 1-morphisms in $\mathcal{S}$ $(f\colon X \to X', f'\colon \mathcal{X} \to \mathcal{X}')$ and a 2-morphism α in $\mathcal{S}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle p}\downarrow & \overset{\alpha}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle p'} \\ \mathcal{X} & \xrightarrow[f']{} & \mathcal{X}' \end{array} .$$

   The composition of 1-morphisms is defined componentwise.

2. Given two 1-morphisms $(f, f', \alpha), (g, g', \beta)\colon (\mathcal{X}, p\colon X \to \mathcal{X}) \to (\mathcal{X}', p'\colon X' \to \mathcal{X}')$, a 2-morphism is by definition a 2-morphism from f' to g' in $\mathcal{S}$.

These data define a 2-category which we denote $\mathcal{RDS}$. We note that we have a forgetful 2-functor

$$For\colon \mathcal{RDS} \to \mathcal{DS}, \quad (\mathcal{X}, p\colon X \to \mathcal{X}) \mapsto \mathcal{X}.$$

5 The 2-category of Lie groupoids

We define the 2-category of Lie groupoids as follows.

0. The objects are the Lie groupoids $(G_0, G_1, s, t, e, m, i)$.

1. A 1-morphim between of Lie groupoids from $(G_0, G_1, s, t, e, m, i)$ to $(G'_0, G'_1, s', t', e', m', i')$ is a pair of smooth morphisms $(f_0\colon G_0 \to G'_0, f_1\colon G_1 \to G'_1)$ which commutes with all the structure maps.

2. Given two 1-morphisms $(f_0, f_1), (g_0, g_1)\colon (G_0, G_1) \to (G'_0, G'_1)$, a 2-morphism $c\colon (f_0, f_1) \Rightarrow (g_0, g_1)$ is a smooth morphism $c\colon G_0 \to G'_1$ such that $s'c = f_0$, $t'c = g_0$ and the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{(g_1, cs)} & G'_1 \underset{s', G'_0, t'}{\times} G'_1 \\ {\scriptstyle (ct, f_1)}\downarrow & & \downarrow{\scriptstyle m'} \\ G'_1 \underset{s', G'_0, t'}{\times} G'_1 & \xrightarrow[m']{} & G'_1 \end{array}$$

commutes. For $(f_0, f_1) = (g_0, g_1)$, the identity 2-morphism is given by $c := ef_0$. Given two 2-morphisms $c\colon (f_0, f_1) \Rightarrow (g_0, g_1)$ and $c'\colon (g_0, g_1) \Rightarrow (h_0, h_1)$, their composition is defined by

$$G_0 \xrightarrow{c', c} G'_1 \underset{s', G'_0, t'}{\times} G'_1 \xrightarrow{m'} G'_1.$$

One checks that the axioms of a 2-category are satisfied. We denote $\mathcal{LG}$ the 2-category defined above.

6 From rigidified differentiable stacks to Lie groupoids

We define a 2-functor $A\colon \mathcal{RDS} \to \mathcal{LG}$ as follows.

0. Let $(\mathcal{X}, p\colon X \to \mathcal{X})$ be an object of $\mathcal{RDS}$. We define

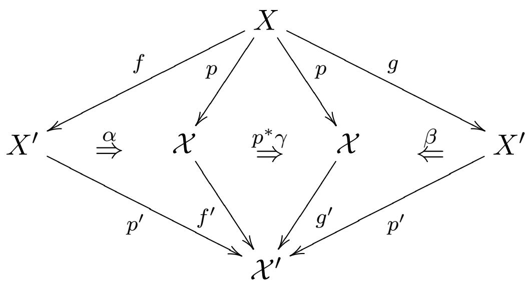
$$\begin{aligned} & G_0 := X, \\ & G_1 := X \underset{p, \mathcal{X}, p}{\times} X, \\ & s := pr_1\colon X \underset{p, \mathcal{X}, p}{\times} X \to X, \\ & t := pr_2\colon X \underset{p, \mathcal{X}, p}{\times} X \to X, \\ & e := (Id_X, Id_X)\colon X \to X \underset{p, \mathcal{X}, p}{\times} X, \quad x \mapsto (x, x), \\ & m := \ (X \underset{p, \mathcal{X}, p}{\times} X) \underset{s, X, t}{\times} (X \underset{p, \mathcal{X}, p}{\times} X) = X \underset{p, \mathcal{X}, p}{\times} X \underset{p, \mathcal{X}, p}{\times} X \xrightarrow{(pr_1, pr_3)} X \underset{p, \mathcal{X}, p}{\times} X \ , \end{aligned}$$

$$i\colon X \underset{p,\mathcal{X},p}{\times} X \to X \underset{p,\mathcal{X},p}{\times} X, \quad (x,y) \mapsto (y,x).$$

Then one checks that $A(\mathcal{X}, p\colon X \to \mathcal{X}) := (G_0, G_1, s, t, e, m, i)$ is a Lie groupoid.

1. If $(f, f', \alpha) : (\mathcal{X}, p\colon X \to \mathcal{X}) \to (\mathcal{X}', p'\colon X' \to \mathcal{X}')$ is a 1-morphism in $\mathcal{RDS}$, we define the 1-morphism $A(f, f', \alpha)$ in $\mathcal{LG}$ by $A(f, f', \alpha) := (f, f \times f)$.

2. Let $(f, f', \alpha), (g, g', \beta)\colon (\mathcal{X}, p\colon X \to \mathcal{X}) \to (\mathcal{X}', p'\colon X' \to \mathcal{X}')$ be two 1-morphisms and a 2-morphism from (f, f', α) to (g, g', β) in $\mathcal{RDS}$. Then by definition, we have a 2-morphism in $\mathcal{S}$, $\gamma\colon f' \to g'$ which induces the 2-morphism in $\mathcal{S}$ $p^*\gamma\colon f' \circ p \Rightarrow g' \circ p$. So we have the following diagram.



We define $A(\gamma) \in \mathrm{Hom}_{smooth}(X, X' \underset{p',\mathcal{X}',p'}{\times} X') = \mathrm{Hom}_{\mathcal{S}}(X, X' \underset{p',\mathcal{X}',p'}{\times} X') = X' \underset{p',\mathcal{X}',p'}{\times} X'\ (X)$ (cf. YLS) by $A(\gamma) = (f, g, \beta^{-1} \star p^*\gamma \star \alpha)$.

7 From Lie groupoids to rigidified differentiable stacks

We define a 2-functor $B'\colon \mathcal{LG} \to \mathcal{RDS}$ as follows.

0. Let $G = (G_0, G_1, s, t, e, m, i)$ be a Lie groupoid. We associate to G the differentiable stack BG (see part 1):

$$\begin{array}{rrcl} BG\colon & \mathcal{M} & \to & \mathcal{G} \\ & M & \mapsto & BG(M) \\ & f\colon M \to N & \mapsto & f^*\colon BG(N) \to BG(M). \end{array}$$

The following lemma gives an atlas for BG.

Lemma 7.1 *[H, Lem 3.1] – The map $u\colon G_0 \to BG$ which corresponds to the unit G-bundle $U(G)$ over G_0 (cf. YLS) is an atlas of BG.*

We define the object $B'G$ of $\mathcal{RDS}$ as $B'G := (BG, u\colon G_0 \to BG)$.

Remark 7.2 – *The inverse map of G, $i\colon G_1 \to G_1$, induces a canonical isomorphism in the category of G-bundles over G_1 from $s^*U(G)$ to $t^*U(G)$ which corresponds (cf. YLS) to a unique 2-morphism ι in $\mathcal{S}$*

$$\iota\colon (u \circ s\colon G_1 \to BG) \Rightarrow (u \circ t\colon G_1 \to BG).$$

Then the triple (s,t,ι) gives a 1-morphism $\sigma\colon G_1 \to G_0 \underset{u,BG,u}{\times} G_0$ which is an isomorphism, and

$$(G_1,\ s\colon G_1 \to G_0,\ t\colon G_1 \to G_0, \iota\colon u\circ s \Rightarrow u \circ t)$$

is the fiber product of $u\colon G_0 \to BG$ and $u\colon G_0 \to BG$ in the 2-category $\mathcal{S}$.

1. Let $(f_0, f_1)\colon (G_0, G_1) \to (G'_0, G'_1)$ be a 1-morphism in $\mathcal{LG}$.

   i) Let $f\colon G_0 \to BG'$ be the composition $u' \circ f_0$ where $u'\colon G'_0 \to BG'$ is the morphism of stacks which corresponds to the unit G'-bundle over G'_0.

   ii) We want now to define a 2-morphism in $\mathcal{S}$

$$\alpha\colon \quad \begin{array}{ccc} (f\circ pr_1\colon G_0 \underset{u,BG,u}{\times} G_0 \to BG') & \Longrightarrow & (f\circ pr_2\colon G_0 \underset{u,BG,u}{\times} G_0 \to BG') \\ \| & & \| \\ (u'\circ f_0 \circ s = u' \circ s' \circ f_1\colon G_1 \to BG') & & (u' \circ f_0 \circ t = u' \circ t' \circ f_1\colon G_1 \to BG'). \end{array}$$

   We define α as $\alpha := f_1^* \iota'$ where

$$\iota'\colon (u'\circ s'\colon G'_1 \to BG') \Rightarrow (u' \circ t'\colon G'_1 \to BG')$$

   is the canonical 2-morphism in $\mathcal{S}$ associated to the inverse map of the Lie groupoid (G'_0, G'_1) (see Remark 7.2).

   iii) Using Proposition 3.1, we associate to (f,α) a canonical pair

$$(\varphi\colon BG \to BG', \beta\colon \varphi\circ u \Rightarrow f)$$

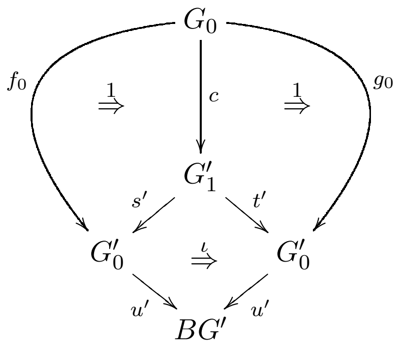
   such that the triple $(f_0, \varphi, \beta^{-1})$ defines a 1-morphism between $B'G$ and $B'G'$ in $\mathcal{RDS}$ which we denote $B'(f_0, f_1)$.

2. Let $(f_0, f_1), (g_0, g_1)\colon (G_0, G_1) \to (G'_0, G'_1)$ be two 1-morphisms and let $c\colon G_0 \to G'_1$ be a 2-morphism from (f_0, f_1) to (g_0, g_1) in $\mathcal{LG}$.

   i) We want to define a 2-morphism γ in $\mathcal{S}$

$$\gamma\colon (u'\circ f_0\colon G_0 \to BG') \Rightarrow (u' \circ g_0\colon G_0 \to BG').$$

   We have $s' \circ c = f_0$ and $t' \circ c = g_0$. Let γ be the composition of the 2-morphisms in the following diagram



   where the 2-morphism ι is defined in Remark 7.2.

ii) If we denote $B'(f_0, f_1) =: (f_0, \varphi, \beta^{-1})$ and $B'(g_0, g_1) =: (g_0, \varphi', \beta'^{-1})$, then using Proposition 3.2, we associate to γ a canonical 2-morphism θ

$$\theta\colon \varphi \Rightarrow \varphi'$$

and we define $B'(c)$ as $B'(c) = \theta$.

8 Comparison of two 2-categories

Theorem 8.1 – *The above 2-functors $A\colon \mathcal{RDS} \to \mathcal{LG}$ and $B'\colon \mathcal{LG} \to \mathcal{RDS}$ are inverse equivalences of 2-categories.*

Proof:

a) We construct an isomorphism of 2-functors $\nu\colon B' \circ A \xrightarrow{\sim} Id_{\mathcal{RDS}}$. Let $(\mathcal{X}, p\colon X \to \mathcal{X})$ be an object of $\mathcal{RDS}$. We need to define a 1-isomorphism, $\nu(\mathcal{X}, p\colon X \to \mathcal{X})$, in $\mathcal{RDS}$

$$\begin{array}{ccc} B'(A(\mathcal{X}, p\colon X \to \mathcal{X})) & \xrightarrow{\nu(\mathcal{X}, p\colon X \to \mathcal{X})} & (\mathcal{X}, p\colon X \to \mathcal{X}). \\ \| & & \\ (B(X, X \underset{p,\mathcal{X},p}{\times} X), u\colon X \to B(X, X \underset{p,\mathcal{X},p}{\times} X)) & & \end{array}$$

We consider the morphism $p\colon X \to \mathcal{X}$ and we want to build a 2-morphism in $\mathcal{S}$

$$\alpha\colon (u \circ pr_1\colon X \underset{u,B(X,X \underset{p,\mathcal{X},p}{\times} X),u}{\times} X \to \mathcal{X}) \Rightarrow (u \circ pr_2\colon X \underset{u,B(X,X \underset{p,\mathcal{X},p}{\times} X),u}{\times} X \to \mathcal{X})$$

to define a 1-morphism $\varphi\colon B(X, X \underset{p,\mathcal{X},p}{\times} X) \to \mathcal{X}$ and a 2-morphism $\beta\colon \varphi \circ u \Rightarrow p$ in $\mathcal{S}$ using Proposition 3.1. By Remark 7.2, the 2-morphisms

$$(p \circ pr_1\colon X \underset{u,B(X,X \underset{p,\mathcal{X},p}{\times} X),u}{\times} X \to \mathcal{X}) \Rightarrow (p \circ pr_2\colon X \underset{u,B(X,X \underset{p,\mathcal{X},p}{\times} X),u}{\times} X \to \mathcal{X})$$

correspond bijectively to the 2-morphisms

$$(p \circ pr_1\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}) \Rightarrow (p \circ pr_2\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}).$$

We define α to be the 2-morphism which corresponds to the canonical 2-morphism

$$\iota\colon (p \circ pr_1\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X}) \Rightarrow (p \circ pr_2\colon X \underset{p,\mathcal{X},p}{\times} X \to \mathcal{X})$$

induced by the inverse map of the Lie groupoid $A(\mathcal{X}, p\colon X \to \mathcal{X})$, i.e. the map which interchanges both factors on $X \underset{p,\mathcal{X},p}{\times} X$ (cf. Remark 7.2). One checks that the cocycle condition holds. By Proposition 3.1, we get a canonical pair

$$(\varphi\colon B(X, X \underset{p,\mathcal{X},p}{\times} X) \to \mathcal{X}, \beta\colon \varphi \circ u \Rightarrow p)$$

as explained above.
It follows from Remark 7.2 and the property $\varphi \circ u \simeq p$ that φ is a monomorphism in $\mathcal{S}$. This is

also an epimorphism (since p is), thus an isomorphism (cf. Lemma 2.6).
We define the 1-isomorphism $\nu(\mathcal{X}, p\colon X \to \mathcal{X})$ as

$$\nu(\mathcal{X}, p\colon X \to \mathcal{X}) := (Id_X,\ \varphi,\ \beta^{-1}).$$

One checks that the data $\nu(\mathcal{X}, p\colon X \to \mathcal{X})$ for $(\mathcal{X}, p\colon X \to \mathcal{X}) \in Ob(\mathcal{RDS})$ define a 2-natural transformation $\nu\colon B' \circ A \to Id_{\mathcal{RDS}}$ which is an isomorphim of 2-functors.

b) We define an isomorphism of 2-functors $\eta\colon A \circ B' \xrightarrow{\sim} Id_{\mathcal{LG}}$. Let $G = (G_0, G_1, s, t, e, m, i)$ be a Lie groupoid. Then we have

$$A \circ B'(G_0, G_1) = (G_0, G_0 \underset{u,BG,u}{\times} G_0)$$

and a canonical ismomorphism $\sigma\colon G_1 \to G_0 \underset{u,BG,u}{\times} G_0$ by Remark 7.2. One checks that

$$\eta(G) := (Id_{G_0}, \sigma^{-1})\colon A \circ B'(G_0, G_1) = (G_0, G_0 \underset{u,BG,u}{\times} G_0) \to (G_0, G_1)$$

is an 1-isomorphism of Lie groupoids and the data $\eta(G)$ for $G \in Ob(\mathcal{LG})$ define a natural transformation $\nu\colon B' \circ A \to Id_{\mathcal{RDS}}$ which is an isomorphim of 2-functors.

□

9 Localization w.r.t. weak equivalences and a comparison of two bicategories

Many properties of Lie groupoids are invariant under Morita equivalences (cf. [M2, 2.4]) (e.g. being proper, see [MM, Part 5.4]). Thus it might be useful to consider 1-morphisms in $\mathcal{LG}$ up to Morita equivalences (cf. [M2, 2.4]). We note that being étale is not invariant under Morita equivalences since any Lie groupoid is Morita equivalent (cf. [M2, 2.4]) to an étale Lie groupoid. On the other it is not so easy to work in the 2-category $\mathcal{RDS}$ where an object is a differentiable stack with a fixed atlas.

We would like to have a ”localized” version of 8.1, i.e. an equivalence of 2-categories between the 2-category ” $\mathcal{LG}$ localized w.r.t. weak equivalences” (cf. [M2, 2.4]) and some 2-category constructed from $\mathcal{RDS}$. In fact we will consider bicategories rather than 2-categories (see Theorem 9.2).

Since any Lie groupoids is Morita equivalent to an étale Lie groupoid and since we would like to invert all the weak equivalences, it is natural to restrict our attention to the full sub-2-category $\mathcal{ELG}$ of $\mathcal{LG}$ such that $Ob(\mathcal{ELG})$ is the class of all étale Lie groupoids. Let W be the class of all weak equivalences between étale Lie groupoids.

Using the notion of weak fiber product [MM, p. 123-124] one shows that any diagram of 1-morphisms of étale Lie groupoids

$$\begin{array}{ccc} & & G' \\ & & \downarrow{\scriptstyle f} \\ G' & \xrightarrow[\varepsilon]{} & H \end{array}$$

where ε is a weak equivalence, can be completed to get a 2-commutative diagram of 1-morphisms of Lie groupoids

$$\begin{array}{ccc} K & \xrightarrow{\varepsilon'} & G' \\ {\scriptstyle f'}\downarrow & \overset{\theta}{\swarrow} & \downarrow{\scriptstyle f} \\ G' & \xrightarrow[\varepsilon]{} & H \end{array} \tag{2}$$

where ε' is a weak equivalence (see [M1, 7.5]). This is a weaker form of a property[1] we ask for in the definition of a class admitting a calculus of fractions, i.e. we can find f', ε' such that the diagram (2) commutes ($\theta = 1$). This leads us to consider bicategories instead of 2-categories.

The class W satisfies the properties $BF1, \dots, BF5$ of [P, 2.1]. In part 4.1 of [P], Pronk checks this for topological stacks. The arguments works as well for the differentiable stacks. Thus we can consider the bicategory of fractions $\mathcal{ELG}[W^{-1}]$ defined in [P, Part 2]).

Proposition 9.1 – *Let $(f_0, f_1)\colon G = (G_0, G_1, s, t, e, m, i) \to G' = (G'_0, G'_1, s', t', e', m', i')$ be a morphism of Lie groupoids and let (f_0, φ, β) be the 1-morphism $B'(f_0, f_1)$ in $\mathcal{RDS}$. Then the 1-morphism $\varphi\colon BG \to BG'$ in $\mathcal{RDS}$ is an isomorphism if and only if (f_0, f_1) is a weak equivalence.*

Proof:

i) Using Proposition 2.7 and Remarks 7.2 and 2.5, one observes that φ is an epimorphism if and only if the map $t' \circ pr_2\colon G_0 \underset{f_0, G'_0, s'}{\times} G'_1 \to G'_0$ is a surjective submersion.

ii) It follows from Remark 7.2 that the following commutative square

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & G'_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s',t')} \\ G_0 \times G_0 & \xrightarrow{(f_0,f_0)} & G'_0 \times G'_0 \end{array}$$

is cartesian if and only if φ is an isomorphism.

iii) Then the result is a consequence of Proposition 2.6.

□

As a consequence, by the universal property of the bicategory $\mathcal{ELG}[W^{-1}]$ (see [P, p. 253]), the 2-functor

$$For \circ B'_{|\mathcal{ELG}}\colon \mathcal{ELG} \xrightarrow{B'_{|\mathcal{ELG}}} \mathcal{RDS} \xrightarrow{For} \mathcal{DS}$$

factors through the canonical morphism of bicategories $U\colon \mathcal{ELG} \to \mathcal{ELG}[W^{-1}]$ and we get a canonical morphism of bicategories

$$B\colon \mathcal{ELG}[W^{-1}] \to \mathcal{DS}$$

such that $For \circ B' = B \circ U$.

The following Theorem is an analog of Theorem 8.1 where one considers the morphisms of Lie groupoids up to Morita equivalences and the differentiable stacks without any fixed atlas.

[1] The author does not know if this stronger property holds for the class W.

Theorem 9.2 – *The morphism of bicategories $B\colon \mathcal{ELG}[W^{-1}] \to \mathcal{DS}$ defines an equivalence of bicategories.*

Proof: We check the necessary and sufficient conditions given in [P, Prop 24].

i) *B is essentially surjective on objects.*

Since B' defines an equivalence of 2-categories, it is essentially surjective on objects. The same property holds for For since any differentiable stack has an atlas. Then the claim follows from the fact that any Lie groupoids is Morita equivalent to an étale Lie groupoid.

ii) *B is fully faithful on 2-cells.*

This is clear for B' since it defines an equivalence of 2-categories and also for For because $\mathcal{DS}$ and $\mathcal{RDS}$ have the same 2-cells.

iii) *For all $G = (G_0, G_1), G' = (G'_0, G'_1) \in Ob(\mathcal{ELG})$, for any 1-morphism $F\colon BG \to BG'$ in $\mathcal{DS}$, there exists a weak equivalence w of Lie groupoids and a 1-morphism $f\colon G \to G'$ in $\mathcal{LG}$ such that $B(f)$ and $F \circ B(w)$ are isomorphic.*

It is a consequence of the following two lemmas (cf. Proposition 9.5).

□

Lemma 9.3 – *Let $G = (G_0, G_1)$, $G' = (G'_0, G'_1)$ be two Lie groupoids and let $F\colon BG \to BG'$ be a 1-morphism in $\mathcal{S}$. Then there exist a Lie groupoid $H = (H_0, H_1)$, a weak equivalence of Lie groupoids $(f_0, f_1)\colon (H_0, H_1) \to (G_0, G_1)$ and a smooth map $f'_0\colon H_0 \to G'_0$ such that the diagram*

$$\begin{array}{ccc} H_0 \xrightarrow{f_0} G_0 & \xrightarrow{u} & BG, \\ {\scriptstyle f'_0}\downarrow & \swarrow & \downarrow{\scriptstyle F} \\ G'_0 & \xrightarrow[u']{} & BG' \end{array}$$

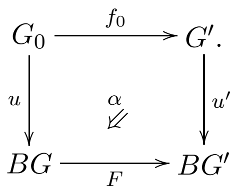
2-commutes. As above, $u\colon G_0 \to BG$ (resp. $u'\colon G'_0 \to BG'$) is the morphism which corresponds to the unit bundle of G (resp. G') (cf. YLS).

Proof: Let H_0 be the manifold $G_0 \underset{F\circ u, BG', u'}{\times} G'_0$. Then we have two canonical smooth maps

$$f_0 := pr_1\colon H_0 \to G_0 \quad \text{and } f'_0 := pr_2\colon H_0 \to G'_0$$

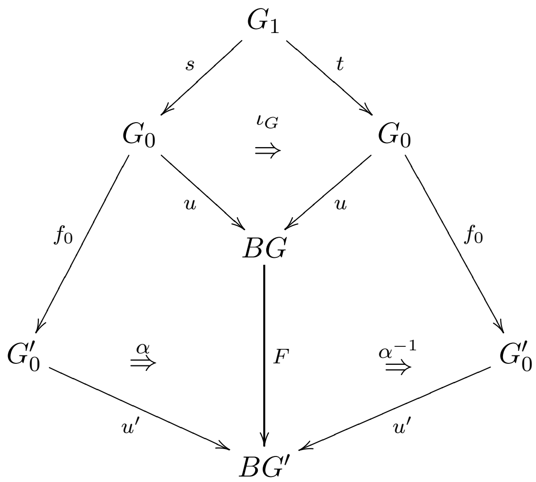
and a canonical 2-morphism $can\colon F \circ u \circ f_0 \Rightarrow u' \circ f'_0$. Moreover f_0 is a surjective submersion. Thus we can consider the induced Lie groupoid $f_0^* G = (H_0, H_1)$ (see [M2, p. 121-122]) and we have a canonical map $f_1\colon H_1 \to G_1$ such that (f_0, f_1) is a morphism of Lie groupoids. In fact one checks easily that (f_0, f_1) is a weak equivalence. □

Lemma 9.4 – *Let $G = (G_0, G_1, s, t, e, m, i)$, $G' = (G'_0, G'_1, s', t', e', m', i')$ be two Lie groupoids and let $F\colon BG \to BG'$ be a morphism in $\mathcal{S}$ such that there exist $f_0\colon G_0 \to G'_0$ and a 2-morphism $\alpha\colon u' \circ f_0 \to F \circ u$, i.e. a 2-commutative diagram*



Then there exists a smooth map $f_1\colon G_1 \to G'_1$ such that (f_0, f_1) is a morphism of Lie groupoids and $B(f_0, f_1) \simeq F$.

Proof: Let β be the composition of the 2-morphisms of the following diagram.



By the universal property of the fiber product in $\mathcal{S}$, $(f_0 \circ s, f_0 \circ t, \beta)$ defines a unique morphism of stacks

$$f_1\colon G_1 \to G'_0 \underset{u', BG', u'}{\times} G'_0 = G'_1$$

such that $f_0 \circ s = s' \circ f_1$ and $f_0 \circ t = t' \circ f_1$ (cf. YLS). One checks that (f_0, f_1) is a morphism of Lie groupoids and that $B'(f_0, f_1)$ is the morphism associated to $(u' \circ f_0, \alpha)$ via Proposition 3.1, i.e. $B(f_0, f_1) \simeq F$.

□

Proposition 9.5 – *Let $G = (G_0, G_1), G' = (G'_0, G'_1)$ be two Lie groupoids and let $F\colon BG \to BG'$ be a 1-morphism in $\mathcal{DS}$. There exists a weak equivalence w of Lie groupoids and a 1-morphism $f\colon G \to G'$ in $\mathcal{LG}$ such that $B(f)$ and $F \circ B(w)$ are isomorphic.*

Proof: We apply Lemma 9.3 to get a Lie groupoid H_0, a weak equivalence of Lie groupoids

$$(f_0, f_1)\colon (H_0, H_1) \to (G_0, G_1)$$

and a smooth map $f'_0\colon H_0 \to G'_0$ such that

$$F \circ u \circ f_0 \simeq u' \circ f'_0. \tag{3}$$

Let us introduce the morphism $v\colon H_0 \to BH$ which corresponds to the unit bundle of H (cf. YLS). We have

$$B(f_0, f_1) \circ v \simeq u \circ f_0. \tag{4}$$

Combining (3) and (4), we get

$$F \circ B(f_0, f_1) \circ v \simeq u' \circ f_0'. \tag{5}$$

By Lemma 9.4 and the relation (5), there exists $f_1'\colon H_1 \to G_1'$ such that (f_0', f_1') is a morphism of Lie groupoids and such that

$$B(f_0', f_1') \simeq F \circ B(f_0, f_1).$$

□

10 The bicategory of Lie groupoids with generalized morphisms

We recall now the definition of a generalized morphism between Lie groupoids.

Definition 10.1 *[LTX, Def 2.1] – Let $G = (G_0, G_1)$ and $G' = (G_0', G_0')$ be two Lie groupoids. A generalized morphism from G to G' is given by a manifold P, two smooth maps*

$$\begin{array}{ccc} P & \xrightarrow{\varepsilon} & G_0' \,, \\ {\scriptstyle\pi}\downarrow & & \\ G_0 & & \end{array}$$

a left action λ of G along π, a right action ρ of G' along ε such that the two actions commute and P is a principal G'-bundle over G_0.

Then we introduce a notion of (iso)morphisms between two generalized morphisms.

Definition 10.2 *[HM, Def 3.1] – Let $(P, \pi, \varepsilon, \lambda, \rho)$ and $(P', \pi', \varepsilon', \lambda', \rho')$ be two generalized morphisms from G to G'. A generalized 2-morphisms from $(P, \pi, \varepsilon, \lambda, \rho)$ to $(P', \pi', \varepsilon', \lambda', \rho')$ is a diffeomorphism $P \to P'$ which is G- and G'-equivariant.*

Remark 10.3 – *One defines in an obvious way the composition of two generalized 2-morphisms between two generalized morphisms between two fixed Lie groupoids. Every generalized 2-morphism is invertible.*

The following two lemmas explain the terminology.

Lemma 10.4 *[LTX, p. 846] – Let $(f_0, f_1)\colon (G_0, G_1, s, t, e, m, i) \to (G_0', G_1', s', t', e', m', i')$ be a 1-morphism of Lie groupoids. Then the following data:*

$$\begin{aligned}
&P := G_0 \underset{f_0, G_0', t'}{\times} G_1', \\
&\pi\colon P \to G_0, \quad (x, g') \mapsto x, \\
&\varepsilon\colon P \to G_0', \quad (x, g') \mapsto s'(g'), \\
&\lambda\colon G_1 \underset{s, G_0, \pi}{\times} P \to P, (g, (x, g')) \mapsto (t(g), f_1(g)g'), \\
&\rho\colon P \underset{\varepsilon, G_0', t}{\times} G_1', ((x, g'), h') \mapsto (s, g'h').
\end{aligned}$$

define a canonical generalized isomorphism which we called the generalized morphism associated to (f_0, f_1).

Proof: Straightforward. $\square$

We define the identity generalized morphism of a Lie groupoid G, denoted by Id_G^+, as the associated generalized morphism associated to $(Id_{G_0}, Id_{G_1})\colon (G_0, G_1) \to (G_0, G_1)$.

Lemma 10.5 *– Let (f_0, f_1) and (g_0, g_1) be two 1-morphisms of Lie groupoids from $(G_0, G_1, s, t, e, m, i)$ to $(G'_0, G'_1, s', t', e', m', i')$. Let $c\colon G_0 \Rightarrow G'_1$ be a 2-morphism in $\mathcal{LG}$. Then the map*

$$G_0 \underset{f_0, G'_0, t'}{\times} G'_1 \to G_0 \underset{g_0, G'_0, t'}{\times} G'_1, \quad (x, g') \mapsto (x, c(x)g')$$

defines a generalized 2-morphism from $C(f_0, f_1)$ to $C(g_0, g_1)$ called the generalized 2-morphism associated to c.

Proof: Straightforward. $\square$

Proposition 10.6 *[LTX, Prop 2.2] – Let $(P, \pi, \varepsilon, \lambda, \rho)$ be a generalized morphism from $(G_0, G_1, s, t, e, m, i)$ to $(G'_0, G'_1, s', t', e', m', i')$ and let $(P', \pi', \varepsilon', \lambda', \rho')$ be a generalized morphism from $(G'_0, G'_1, s', t', e', m', i')$ to $(G''_0, G''_1, s'', t'', e'', m'', i'')$. The following data*

$$\begin{aligned}
&P'' := \text{ the quotient of } P \underset{\varepsilon, G'_0, \pi'}{\times} P' \text{ by the following action of } G'_1 : \quad (p, p')g' = (pg', (g')^{-1}p'),\\
&\pi''\colon P'' \to G'_0, \quad [(p, p')] \mapsto \varepsilon(p) = \pi'(p'),\\
&\varepsilon''\colon P'' \to G''_0, \quad [(p, p')] \mapsto \varepsilon'(p'),\\
&\lambda\colon G'_1 \underset{s', G'_0, \pi''}{\times} P'' \to P'', \quad (g', [(p, p')]) \mapsto [(p, g'p')] = [(p(g')^{-1}, p'],\\
&\rho''\colon P'' \underset{\varepsilon'', G'_0, t'}{\times} G''_1 \to P'', \quad ([(p, p')], g'') \mapsto [(p, p'g'')].
\end{aligned}$$

define a generalized morphism from $(G_0, G_1, s, t, e, m, i)$ to $(G''_0, G''_1, s'', t'', e'', m'', i'')$ which is by definition the composition of $(P, \pi, \varepsilon, \lambda, \rho)$ and $(P', \pi', \varepsilon', \lambda', \rho')$.

Remark 10.7 *[HM, below Def 3.1] – Since we define the composition of generalized morphisms using a pullback, this composition is not associative. In fact, it is easy to check that the composition of generalized morphisms is associative up to generalized 2-isomorphisms.*

We define the bicategory of étale Lie groupoids with generalized isomorphisms which we denote $\mathcal{ELG}^+$ as follows.

0. The objects are the étale Lie groupoids.
1. The 1-morphisms are the generalized 1-morphisms.
2. The 2-morphisms are the generalized 2-morphisms.

11 Localization w.r.t. weak equivalences versus generalized morphisms

We define a morphism of bicategories $C'\colon \mathcal{ELG} \to \mathcal{ELG}^+$ as follows.

0. C' is the identity on the 0-cells.

1. If f is a 1-morphism of étale Lie groupoids then $C'(f)$ is the generalized morphism associated to (f_0, f_1) (cf. Lemma 10.4).

2. If c is a 2-morphism in $\mathcal{ELG}$ then $C'(f)$ is the generalized 2-morphism associated to c (cf. Lemma 10.5).

Our first aim is to check that the morphism of bicategories $C'\colon \mathcal{ELG} \to \mathcal{ELG}^+$ factors through the canonical morphism of bicategories $U\colon \mathcal{ELG} \to \mathcal{ELG}[W^{-1}]$.

Proposition 11.1 –

i) Let $f := (P, \pi, \varepsilon, \lambda, \rho)$ be a generalized morphism from $G = (G_0, G_1, s, t, e, m, i)$ to $G' = (G'_0, G'_1, s', t', e', m', i')$ such that $(P, \pi, \varepsilon, \lambda)$ is a (left) principal G-bundle over G'_0. Then there exists a generalized morphism $g\colon G' \to G$ such that $g \circ f = Id^+_G$ and $f \circ g \simeq Id^+_{G'}$.

ii) Let $f\colon G \to G'$ be a weak equivalence of Lie groupoids. Then there exists a generalized morphism $g\colon G' \to G$ such that $g \circ C'(f) = Id^+_G$ and $C'(f) \circ g \simeq Id^+_{G'}$.

Proof:

i) If $(P, \pi, \varepsilon, \lambda, \rho)$ is a (left) principal G-bundle over G'_0 then the following data

$$\begin{aligned}
&P,\\
&\pi' := \varepsilon\colon P \to G'_0,\\
&\varepsilon' := \pi\colon P \to G_0,\\
&\lambda'\colon G'_1 \underset{s', G'_0, \varepsilon}{\times} P \to P, \quad (g', p) \mapsto p(g')^{-1},\\
&\rho'\colon P \underset{\pi, G_0, t}{\times} G_1 \to P, \quad (p, g) \mapsto g^{-1}p.
\end{aligned}$$

define a generalized morphism from G' to G which we denote g. One checks easily that $g \circ f = Id^+_G$ and $f \circ g \simeq Id^+_{G'}$.

ii) Let $f\colon G \to G'$ be a weak equivalence of Lie groupoids and let $C'(f) = (P, \pi, \varepsilon, \lambda, \rho)$. Then one Remarks that $(P, \pi, \varepsilon, \lambda)$ is a (left) principal G-bundle over G'_0.

□

By the universal property of $\mathcal{ELG}[W^{-1}]$ (cf. [P, p. 253]), we get a canonical morphism of bicategories $C\colon \mathcal{ELG}[W^{-1}] \to \mathcal{ELG}^+$ such that $C \circ U = C'$.

Theorem 11.2 – *The morphism of bicategories $C\colon \mathcal{ELG}[W^{-1}] \to \mathcal{ELG}^+$ induces an equivalence of bicategories.*

Proof: We check the necessary and sufficient conditions given in [P, Prop 24].

i) *C is essentially surjective on objects.*
Straightforward.

ii) *C is fully faithful on 2-cells.*
It follows from the proof of Lemma 2.4 in [LTX].

iii) *For all* $G = (G_0, G_1), G' = (G'_0, G'_1) \in Ob(\mathcal{ELG})$, *for any generalized morphism* $F\colon G \to G'$, *there exists a weak equivalence* w *of Lie groupoids and a 1-morphism* $f\colon G \to G'$ *in* $\mathcal{ELG}$ *such that* $B(f)$ *and* $F \circ B(w)$ *are isomorphic.*
We refer the reader to the proof of Proposition 2.3 in [LTX].

□

12 Comparison of two categories

Let $\mathcal{B}$ be a bicategory. The we define the category $\mathcal{B}_{/2}$ as follows.

0. $Ob(\mathcal{B}_{/2}) = Ob(\mathcal{B})$.

1. Let A, B be two objects in $Ob(\mathcal{B}_{/2})$. Let $\sim$ be the equivalence relation on $\mathrm{Hom}^1_{\mathcal{B}}(A, B)$ defined by $f \sim g$ if and only if there exists a 2 morphisms θ such that $\theta(f) = g$. Then we define the morphisms from A to B in $\mathcal{B}_{/2}$ as

$$\mathrm{Hom}_{\mathcal{B}_{/2}}(A, B) := \mathrm{Hom}^1_{\mathcal{B}}(A, B)/\sim .$$

If $F\colon \mathcal{B} \to \mathcal{C}$ is a morphism of bicategories, then it induces a canonical functor denoted by $F_{/2}$ from $\mathcal{B}_{/2}$ to $\mathcal{C}_{/2}$. Moreover if F is an equivalence of bicategories, then $F_{/2}$ is an equivalence of categories.

Applying this to $B\colon \mathcal{ELG}[W^{-1}) \to \mathcal{DS}$ we get the following theorem.

Theorem 12.1 – *The functor* $B_{/2}\colon (\mathcal{ELG}[W^{-1}])_{/2} \to \mathcal{DS}_{/2}$ *defines an equivalence of categories.*

Remarks 12.2 – *This theorem is not so good to work with stacks. We have lost many informations. For example we can not consider anymore the gluing data on the level of stacks. We refer the reader to the pages 2490 and 2491 of [HM] for a longer discussion.*

We can also apply this general construction to $C\colon \mathcal{ELG}[W^{-1}] \to \mathcal{ELG}^+$. Then we get an equivalence of categories $C_{/2}\colon (\mathcal{ELG}[W^{-1}])_{/2} \to (\mathcal{ELG}^+)_{/2}$. In [LTX], the authors consider the category $(\mathcal{ELG}^+)_{/2}$ and the following diagram links this one and the category $\mathcal{DS}_{/2}$:

$$(\mathcal{ELG}^+)_{/2} \xleftarrow[1:1]{C_{/2}} (\mathcal{ELG}[W^{-1}])_{/2} \xrightarrow[1:1]{B_{/2}} \mathcal{DS}_{/2} .$$

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