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The elliptic polylogarithm I and II
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INTRODUCTION

The aim of these two talks is to introduce an elliptic analogue of Zagier's conjecture. We follow the article of J. Wildeshaus [W3].

In the first one, we define a formalism for an elliptic polylogarithm in a general setting. Then we prove that such an elliptic polylogarithm exists in the context of admissible and graded polarizable variations of mixed Hodge structures (VMHS). After having explained the notion of polylogarithm value at torsion points of a complex elliptic curve, we give an expression for this in terms of Eisenstein-Kronecker series.

In the second talk, we state an elliptic analogue of Zagier's conjecture. Broadly speaking, it gives a recipe for contructing nonzero elements in specific motivic cohomology groups attached to an elliptic curve, and predicts that their images under Beilinson's regulator are polylogarithms. We explain how this is implied by the existence of a suitable category of smooth motivic sheaves, admitting amongst other things the formalism of an elliptic polylogarithm.

1. THE ELLIPTIC POLYLOGARITHM I - HODGE REALIZATION

1.1. **A formalism for an elliptic polylogarithm.** In this section, we define the notion of a formalism for an elliptic polylogarithm as a system of data which satisfies seven axioms. We note that there is another but equivalent way to define the elliptic polylogarithm in the l-adic or Hodge context (e.g. [Ki2, 1.1]). Our approach is useful for discussing Zagier's conjecture for elliptic curves.

We fix some data. Let S be a connected base scheme and F a field of characteristic 0. For each quasi projective and smooth scheme B over S, we have a F-linear abelian category $\mathcal{T}(B)$ with an associative, commutative and unitary (we write $F(0)$ for the neutral element) tensor product, such that $B \mapsto \mathcal{T}(B)$ is natural in a contravariant way.

These data satisfy the following seven axioms.
(A) For B connected, $\mathcal{T}(B)$ is a neutral abelian F-linear Tannakian category and for $f : B_1 \to B_2$, $f^* : \mathcal{T}(B_2) \to \mathcal{T}(B_1)$ is exact.
(B) $\mathcal{T}(B)$ is a tensor category with weights ([W3, Def. 2.4]).
(C) There is an object $F(1)$ of rank 1 and weight -2 in $\mathcal{T}(S)$. For a scheme B over S, we still denote by $F(1)$ the pullback of $F(1)$ by the structural morphism of B. For $V \in \mathcal{T}(B)$, $V(1) := V \otimes F(1)$.
(D) For any elliptic curve $\pi : \mathcal{E} \to B$, there exist an object $R^1\pi_*F$ of rank 2 and weight 1 in $\mathcal{T}(B)$, and an isomorphism $\cup_\pi : \overset{2}{\Lambda} R^1\pi_*F \overset{\sim}{\to} F(-1)$ (the dual of $F(1)$) compatible with base change.

(E) For any elliptic curve $\pi : \mathcal{E} \to B$, we have a morphism $[\ ] : \mathcal{E}(B) \otimes_{\mathbb{Z}} F \to \mathrm{Ext}^1_{\mathcal{T}(B)}(F(0), R^1\pi_* F(1))$.

(F) For any elliptic curve $\pi : \mathcal{E} \to B$, form the base change
$pr_1 : \mathcal{E} \times_B \mathcal{E} \to \mathcal{E}$, and consider $[\Delta] \in \mathrm{Ext}^1_{\mathcal{T}(\mathcal{E})}(F(0), \pi^* R^1\pi_* F(1))$ and $\mathrm{Sym}^{N-1}[\Delta]$. $(\mathrm{Sym}^{N-1}[\Delta])_{N \geq 1}$ forms a projective system (the transition morphisms are induced by $[\Delta] \to F(0)$), the logarithmic pro-sheaf $\mathcal{L}og$ on $\mathcal{E}$. Moreover there exists a projective system (the (small) polylogarithmic extension)
$$\mathcal{P}ol = (\mathcal{P}ol^N)_{N \in \mathbb{N}} \in \varprojlim_{N \in \mathbb{N}} \mathrm{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}(\pi^* R^1\pi_* F(1)_{|\widetilde{\mathcal{E}}}, \mathrm{Sym}^{N-1}[\Delta](1)_{|\widetilde{\mathcal{E}}})$$
where $\widetilde{\mathcal{E}}$ is the complement of the zero section in $\mathcal{E}$, such that
$$\mathcal{P}ol^1 \in \mathrm{Ext}^1_{\mathcal{T}(\widetilde{\mathcal{E}})}(F(0), (\pi^* R^1\pi_* F(1)_{|\widetilde{\mathcal{E}}})^{\vee}(1))$$
coincides with the pushout of $[\Delta]_{|\widetilde{\mathcal{E}}}$ under the isomorphism induced by $\cup_\pi$,
$R^1\pi_* F(1)_{|\widetilde{\mathcal{E}}} \xrightarrow{\sim} (R^1\pi_* F(1)_{|\widetilde{\mathcal{E}}})^{\vee}(1)$, $v \mapsto \cup_\pi(v, \ )$.

(G) Let $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ be an isogeny between two elliptic curves over B. Then $\psi^* \mathcal{L}og_{\mathcal{E}_2} = \mathcal{L}og_{\mathcal{E}_1}$.

1.2. An elliptic polylogarithm for VMHS. We consider the following data. Let $S := \mathrm{Spec}(\mathbb{C})$, $F = \mathbb{Q}$ or $\mathbb{R}$, $\mathcal{T}(B) := \mathrm{VMHS}_F(B)$. Then $f : B_1 \to B_2$ induces $f^* : \mathrm{VMHS}_F(B_2) \to \mathrm{VMHS}_F(B_1)$ given by the pullback at the level of local systems. We explain briefly why the axioms (A) - (G) are fulfilled in this setting.

(A) Let $b \in B(\mathbb{C})$. To $\mathbb{V} \in \mathrm{VMHS}_F(B)$, one associates $\overline{\mathbb{V}}_b$, where $\overline{\mathbb{V}}$ is the local system underlying to $\mathbb{V}$. This defines a fibre functor.

(B) $\mathbb{V} \in \mathrm{VMHS}_F(B)$ has a weight filtration compatible with $\otimes$.

(C) $F(1) \in \mathrm{MHS}_F$ is first Tate's twist.

(D) Consider the (topological) first higher direct image under π of the constant sheaf F on $\mathcal{E}(\mathbb{C})$. Its fibre at $b \in B(\mathbb{C})$ is $H^1(\mathcal{E}_b(\mathbb{C}), F)$. By classical Hodge theory, $H^1(\mathcal{E}_b(\mathbb{C}), F)$ is equipped with a pure Hodge structure of weight 1. The collection $(H^1(\mathcal{E}_b(\mathbb{C}), F))_{b \in B(\mathbb{C})}$ forms an object of $\mathrm{VMHS}_F(B)$ of rank 2 and weight 1 which is, by definition, $R^1\pi_* F$. $\cup_\pi$ is induced by the fibrewise cup product.

(E) The map $[\ ]$ is constructed by using Saito's theory of mixed Hodge modules ([W3, 3.2]).

(F) Let $\widetilde{\pi} : \widetilde{\mathcal{E}} \to B$ be the restriction of π to $\widetilde{\mathcal{E}}$. A fundamental property ([Ki2, Prop. 1.1.3 b)]) of the $\mathcal{L}og$ pro-sheaf is
$$R^n\widetilde{\pi}_* \, \mathcal{L}og \ = \begin{cases} \prod_{k>0} \mathrm{Sym}^k(R^1\pi_* F(1)) \, (-1) & \text{if } n = 1 \\ 0 & \text{else.} \end{cases}$$
Now, the Leray spectral sequence for $\mathrm{RHom}((R^1\pi_* F(1)), \ ) \, o \, R\widetilde{\pi}_*$ and weight considerations give an isomorphism :
$Res : \mathrm{Ext}^1_{\widetilde{\mathcal{E}}}(\widetilde{\pi}^* R^1\pi_* F(1), R^1\widetilde{\pi}_* \, \mathcal{L}og(1)) \xrightarrow{\sim} \mathrm{Hom}_B(R^1\pi_* F(1), R^1\pi_* F(1))$.

We define $\mathcal{P}ol$ by $Res(\mathcal{P}ol) = Id$. The compatibility between $\mathcal{P}ol^1$ and $[\Delta]$ is satisfied ([W2, Prop. 2.4, Prop. 2.5]).

(G) For the compatibility of $\mathcal{L}og$ with respect to isogenies, we refer to [BL, 1.2.10.(vi)].

Remark 1. : *The fibre of $\mathcal{L}og$ at $x \in \widetilde{\mathcal{E}}(\mathbb{C})$ can be described in terms of relative homology groups (*[Le, 2.4.4]*).*

1.3. **Values of the polylogarithm at torsion points (real coefficients).** Let $(E/\mathbb{C}, 0)$ be an elliptic curve over $\mathbb{C}$, x a nonzero torsion point of E and $F = \mathbb{R}$. In this context, $R^1\pi_* F(1) = H^1(E(\mathbb{C}), \mathbb{R})(1) =: H_1$. First, we precise the notion of value of the polylogarithm at x. It is obvious that $0^*\mathcal{L}og = \prod_{k \geq 0} \mathrm{Sym}^k H_1$. Using (G), we prove $x^*\mathcal{L}og = \prod_{k \geq 0} \mathrm{Sym}^k H_1$. So $x^*\mathcal{P}ol$ lies in $\mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), \prod_{k \geq 0} \mathrm{Sym}^k H_1 \otimes H_1^\vee (1))$. The k-th value of $\mathcal{P}ol$ at x, $[x^*\mathcal{P}ol]^k$, is the pushout of $x^*\mathcal{P}ol$ under the composition of the contraction map $\prod_{k \geq 0} \mathrm{Sym}^k H_1 \otimes H_1^\vee (1) \to \prod_{k \geq 0} \mathrm{Sym}^{k-1} H_1(1)$ with the projection on the $(k-2)$-th factor. So $[x^*\mathcal{P}ol]^k \in \mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), \mathrm{Sym}^{k-2} H_1 (1))$.

Let $V \in MHS_\mathbb{R}$, V of weight ≤ -1. Then we have an isomorphism

$$\overline{V \otimes \mathbb{R}(-1)} \xrightarrow{\sim} \overline{V}_\mathbb{C}/\overline{V} \xrightarrow{\sim} \mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), V)$$

which associates to $h \in \overline{V \otimes \mathbb{R}(-1)} \subset \overline{V}_\mathbb{C}$ the following 1-extension : we put the diagonal weight and Hodge filtrations on $\mathbb{C} \oplus \overline{V}_\mathbb{C}$ and we take $< 1-h, \overline{V} >_\mathbb{R} \subset \mathbb{C} \oplus \overline{V}_\mathbb{C}$ as real structure. Applying this result to $V = \mathrm{Sym}^{k-2} H_1 (1)$ one identifies $\mathrm{Sym}^{k-2}\overline{H_1}$ and $\mathrm{Ext}^1_{MHS_\mathbb{R}}(\mathbb{R}(0), \mathrm{Sym}^{k-2} H_1(1))$.

Now, we introduce the Eisenstein-Kronecker series. Fix an isomorphism $\eta : E(\mathbb{C}) \to \mathbb{C}/L$ where L is a lattice in $\mathbb{C}$ and let $\omega(L) := \eta^* dz$. Recall the definition of the Pontryagin product : $(z, \gamma)_L = \exp(\pi.Vol(L)^{-1}.(z\overline{\gamma} - \overline{z}\gamma))$, $z \in \mathbb{C}/L$, $\gamma \in L$. The Eisenstein-Kronecker series $K_{a,b,L} : \mathbb{C} - L \to \mathbb{C}$, for $a, b \geq 1$ is defined by

$$K_{a,b,L}(z) := \sum_{\gamma \in L - \{0\}} \frac{(z, \gamma)_L}{\gamma^a \overline{\gamma}^b}.$$

We are now able to give an explicit formula for $[x^*\mathcal{P}ol]^k$ viewed as an element of $\mathrm{Sym}^{k-2}\overline{H_1}$.

Theorem. [W2, Prop. 1.3, Cor. 4.10 (a)]

(1) *For $k \geq 2$, $G_{E,k}(x) := \sum_{a+b=k-2} K_{a+1,b+1,L}(\eta(x)) \, \omega(L)^a \overline{\omega(L)}^b$, which is an element of $\mathrm{Sym}^{k-2}\overline{H_1}_\mathbb{C}$, lies actually in $\mathrm{Sym}^{k-2}\overline{H_1}$ and does not depend on any choice.*

(2) $[x^*\mathcal{P}ol]^k = G_{E,k}(x)$.

2. The elliptic polylogarithm II - Zagier's conjecture

We begin by stating the so-called weak version of Zagier's conjecture for elliptic curves. It is of inductive nature : there is a statement for each $k \geq 2$, and the k-th step can only be formulated if all previous steps are true.

We will use the following notation for motivic cohomology. For any scheme X and any integers $i, j \in \mathbb{Z}$, we put

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) := K^{(j)}_{2j-i}(X),$$

the j-th Adams eigenspace of Quillen's K-group tensorized with $\mathbb{Q}$.

Let K be a number field and B a smooth, quasi-projective, connected scheme over K or $\mathcal{O}_K$. Let $\pi : \mathcal{E} \to B$ be an elliptic curve. For any integer $k \geq 2$, we wish to construct explicit elements in

$$H^{k-1}_{\mathcal{M}}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}},$$

where $\mathcal{E}^{(k-2)} := \ker(\sum : \mathcal{E}^{k-1} \to \mathcal{E})$ and the subscript $(\cdot)_{\mathrm{sgn}}$ denotes the signature-eigenspace determined by the action of the symmetric group $\mathcal{S}_{k-1}$ on $\mathcal{E}^{(k-2)}$.

For any $k \geq 1$, let $\mathcal{L}^\sharp_k$ be the $\mathbb{Q}$-vector space with basis elements $(\{s\}^\sharp_k, s \in \mathcal{E}(B), s \neq 0)$. Let

$$\phi_1 : \mathcal{L}^\sharp_1 \to \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$\{s\}^\sharp_1 \mapsto s \otimes 1.$$

Put $\mathcal{L}_1 := \mathcal{L}^\sharp_1 / \ker \phi_1 \cong \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ and define $\{s\}_1 := $ class of $\{s\}^\sharp_1 = s \otimes 1$.

Conjecture. *There exist quotients $\mathcal{L}_k$ of $\mathcal{L}^\sharp_k$ (for all $k \geq 2$) with the following properties. Denoting the class of $\{s\}^\sharp_k$ in $\mathcal{L}_k$ by $\{s\}_k$, we define the differential*

$$d_k : \mathcal{L}^\sharp_k \to \mathcal{L}_{k-1} \otimes_{\mathbb{Q}} \mathcal{L}_1$$
$$\{s\}^\sharp_k \mapsto \{s\}_{k-1} \otimes \{s\}_1.$$

Then there exists a homomorphism

$$\phi_k : \ker d_k \to H^{k-1}_{\mathcal{M}}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$$

such that

(1) *ϕ_k is compatible with base change $B' \to B$ and with isogenies $\psi : \mathcal{E}_1 \to \mathcal{E}_2$ satisfying $\ker \psi \subset \mathcal{E}_1(B)$.*
(2) *($B = \mathrm{Spec}\, K$). Let r_∞ be the regulator of Deligne and Beilinson*

$$r_\infty : H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}} \to \left(\bigoplus_{\sigma:K \hookrightarrow \mathbb{C}} \mathrm{Sym}^{k-2} H_{\mathrm{B}}^1(\mathcal{E}^\sigma(\mathbb{C}), \mathbb{R}(1)) \right)^+ ,$$

where H_{B}^1 indicates Betti cohomology and $(\cdot)^+$ denotes the fixed subspace for the complex conjugation acting both on the set $\{\sigma : K \hookrightarrow \mathbb{C}\}$ and on the coefficients $\mathbb{R}(1)$. Then

$$r_\infty(\phi_k(S)) = \left(G_{\mathcal{E}^\sigma(\mathbb{C}), k}(S^\sigma) \right)_\sigma \qquad \text{for all } S \in \ker d_k,$$

where $G_{\mathcal{E}^\sigma(\mathbb{C}), k}(S^\sigma)$ is defined by linearity.

(3) *$(B = \mathrm{Spec} K)$. This condition, which we do not give explicitly here, is an integrality criterion. It gives a necessary and sufficient condition on $S \in \ker d_k$ in order that $\phi_k(S)$ belongs to the integral subspace of motivic cohomology (this $\mathbb{Q}$-subspace is defined using the Néron model of $\mathcal{E}$).*

If the conjecture at step k is true, define $\mathcal{L}_k := \mathcal{L}_k^\sharp / \ker \phi_k$ and go to step $k+1$.

Remark 2. *Condition (2) ensures that the homomorphism ϕ_k isn't trivial.*

Remark 3. *If $s \in \mathcal{E}(B)$ is a nonzero torsion point of $\mathcal{E}$, then $\{s\}_k^\sharp$ belongs to $\ker d_k$, as it can be seen from the definition of $\mathcal{L}_1$. Thus (rational) torsion points of $\mathcal{E}$ always yield elements of the motivic cohomology group of interest.*

Remark 4. *We have the following chain of inclusions*

$$\ker \phi_k \subset \ker d_k \subset \mathcal{L}_k^\sharp.$$

The group on the left should come from the functional equations of the elliptic polylogarithm. The quotient $\ker d_k / \ker \phi_k$ can be identified, via ϕ_k, with a subspace of $H_{\mathcal{M}}^{k-1}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$. This subspace is strict in general (there are elliptic curves with trivial Mordell-Weil group).

We now briefly indicate how the formalism of an elliptic polylogarithm allows us to interpret the conjecture in a convenient way.

By Jannsen's lemma [Ja, Lemma 9.2], the target space of the regulator map r_∞ is given by the $(\cdot)^+$ part of the following 1-extension group

$$\mathrm{Ext}^1_{\mathrm{VMHS}(B)}(\mathbb{R}(0), \mathrm{Sym}^{k-2} V_{2,\mathbb{R}}(1))$$

where $V_{2,\mathbb{R}} := R^1 \pi_* \mathbb{R}(1)$ is pure of weight -1 and rank 2. We hope that the motivic cohomology groups we are interested in are described by similar Ext^1-groups in a suitable category $\mathcal{T}(B)$ of smooth motivic sheaves over B. More precisely, we require that

(1) $\mathcal{T}(B)$ satisfies axioms (A) - (G).

(2) Put $V_2 := R^1\pi_*\mathbb{Q}(1)$. Then, there is a canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{T}(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2}V_2(1)) \cong H^{k-1}_{\mathcal{M}}(\mathcal{E}^{(k-2)}, \mathbb{Q}(k-1))_{\mathrm{sgn}}$$

which is compatible with base change $B' \to B$ and with isogenies $\mathcal{E}_1 \to \mathcal{E}_2$.

(3) There is an (exact) Hodge realization $\mathcal{T}(B) \to \mathrm{VMHS}(B \otimes_{\mathbb{Q}} \mathbb{C})$ which is compatible with axioms (A) - (G) and with r_∞.

Since we work in a category with the formalism of an elliptic polylogarithm, we use the existence of $\mathcal{P}ol \in \mathcal{T}(\widetilde{\mathcal{E}})$ to construct the map ϕ_k. For any section $s \in \mathcal{E}(B)$, $s \neq 0$, we consider $s^*\mathcal{P}ol \in \mathcal{T}(B)$. Using the Tannakian formalism, it turns out that suitable formal linear combinations of $s^*\mathcal{P}ol$ (varying s) yield extensions in $\mathrm{Ext}^1_{\mathcal{T}(B)}(\mathbb{Q}(0), \mathrm{Sym}^{k-2}V_2(1))$. In order to carry out this task, one considers the graded $\mathbb{Q}$-vector space underlying $s^*\mathcal{P}ol$. It is equipped with a Lie algbera representation which can be described by a pro-matrix. The coefficients of the latter give the desired extensions.

Thus the existence of a "good" category of smooth motivic sheaves implies the elliptic Zagier conjecture. For the details we refer to [W3].

Finally we give the known results on the conjecture. We restrict to the case where $\mathcal{E}$ is an elliptic curve defined over a number field K. In the case $k = 2$, the weak version of Zagier's conjecture is already proved in [W3]. In the case $k = 3$ and $K = \mathbb{Q}$, it has been proved by Goncharov and Levin [GL], together with a certain surjectivity property of ϕ_k. In the case where $k = 3$ and K is any number field, the conjecture and the surjectivity property have been proved by Rolshausen and Schappacher [RS].

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