誘導函數ヲ有セザル連續函數ノ簡單ナル例

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T. TAKAGI:—A SIMPLE EXAMPLE OF THE CONTINUOUS FUNCTION WITHOUT DERIVATIVE.

Taking the independent variable t (which for brevity's sake shall be confined to the interval $0\cdots1$) in the form

$$(1)\qquad t=\sum^{n}\frac{c_n}{2^n}\qquad\qquad c_n=0 \text{ or } 1.$$

and putting

$$\tau_n=\frac{c_n}{2^n}+\frac{c_{n+1}}{2^{n+1}}+\frac{c_{n+2}}{2^{n+2}}+\cdots\cdots$$

$$\tau'_n=\frac{1}{2^{n-1}}-\tau_n$$

I define $f(t)$ by

$$f(t)=\sum_{1,\infty}^{n}\gamma_n$$

where $\gamma_n=\tau_n$ or τ'_n according as $c_n=0$ or 1, so that

$$f(t)=\sum\frac{a_n}{2^n}:$$

$$\begin{aligned} a_n&=\nu_n \quad \text{for} \quad c_n=0,\\ &=\pi_n \quad \text{for} \quad c_n=1, \end{aligned}$$

π_n, ν_n denoting the number of o's and 1's resp. among $a_1, a_2, \cdots\cdots a_n$, so that $\pi_n+\nu_n=n$.

Evidently $f(t)$ is one-valued and continuous for $0\leqq t\leqq1$; it is only to be remarked, that the rational numbers of the form $t=\frac{m}{2^n}$

and only these admit of two different representations of the form (1), which however give rise to the same value of $f(t)$, as may be easily verified.

In order to prove the non-existence of the derivative, we calculate $\frac{\Delta f}{\Delta t}$ for special values of Δt :

I. $\qquad c_n=0. \qquad \Delta t=\frac{1}{2^n}$

$$\frac{\Delta f}{\Delta t}=\pi_n-\nu_n-2^{n+1}\tau_{n+1}$$

II. $\qquad c_{n-1}=0, \qquad c_n=\overset{1}{\cancel{0}}; \qquad \Delta t=\frac{1}{2^n}$

$$\frac{\Delta f}{\Delta t}=\pi_n-\nu_n$$

If then $c_n=0$, $c_{n+1}=0$, then the values of $\frac{\Delta f}{\Delta t}$ for $\Delta t=\frac{1}{2^n}$ and for $\Delta t=\frac{1}{2^{n+1}}$ differ from one another by $1-2^{n+1}\,\tau_{n+2}$; and if $c_n=0$, $c_{n+1}=1$, the values of $\frac{\Delta f}{\Delta t}$ for $\Delta t=\frac{1}{2^n}$ and $\Delta t=\frac{1}{2^{n+1}}$ differ by $2^{n+1}\tau_{n+2}$.

From this we easily infer that the fluctuation of $\frac{\Delta f}{\Delta t}$, even if we confine ourselves to the values of Δt, which are of the form $\frac{1}{2^n}$, is finite however large n may be taken.